

Bohr's Theorem for Monogenic Power Series

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Abstract

The main goal of this paper is to generalize Bohr's phenomenon from complex one-dimensional analysis to higher dimensions in the framework of Quaternionic Analysis.

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1 Introduction

In 1914 Bohr discovered that there exists a radius $r \in (0, 1)$ such that if a power series of a holomorphic function converges in the unit disk and its sum has a modulus less than 1, then for $|z| < r$ the sum of the absolute values of its terms is again less than 1. This radius does not depend on the function.

Theorem 1.1 (Bohr, 1914) *Let f be a bounded analytic function in the open unit disk, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk and with modulus less than 1. Then $\sum_{n=0}^{\infty} |a_n| r^n < 1$ for $0 \leq r < \frac{1}{3}$.*

This inequality known as Bohr's inequality is true for $0 \leq r < \frac{1}{3}$ and the constant $\frac{1}{3}$ cannot be improved.

Originally, this theorem was proved for $0 \leq r < \frac{1}{6}$ but soon improved to the sharp result. In Bohr's paper [20] his own proof was published as well as a proof by Wiener based on function theory methods. Later, S. Sidon gave a different proof (see [22]).

Recently, several papers were published, generalizing Bohr's theorem to functions of n complex variables (see [8], [21], [1]). Using the standard multi-index notations $\underline{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\underline{\alpha}| = \alpha_1 + \dots + \alpha_n$, $z := (z_1, \dots, z_n)$, $z_i \in \mathbb{C}$, $z^{\underline{\alpha}} := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$, it is shown in [21] that if a power series $\sum_{\underline{\alpha}} c_{\underline{\alpha}} z^{\underline{\alpha}}$ has a modulus less than 1 in the unit polydisc $\{(z_1, \dots, z_n) : \max_{1 \leq j \leq n} |z_j| < 1\}$, then the sum of the moduli of the terms is less than 1 in the polydisc of radius $\frac{1}{3\sqrt{n}}$.

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In [15], the result shows the possibility to obtain a Bohr type theorem for monogenic functions in the ball in the Euclidean space \mathbb{R}^3 with the additionally condition $f(0) = 0$. It is shown that for $r < 0.047$, the inequality is satisfied. The main purpose of this paper is to check if this theorem can be extended to all monogenic functions with $|f(\mathbf{x})| < 1$ in $B_1(0)$.

Having in mind the analogy to the one-dimensional complex function theory we want to know if the result can be proved for a ball in the Euclidean space and not for a polydisc. It is not the goal here to find a sharp estimate for the most general class of functions.

2 Preliminaries

Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of the Euclidean vector space \mathbb{R}^4 . The vector \mathbf{e}_0 is the scalar unit while the generalized imaginary units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ satisfy the following multiplication rules

$$\begin{aligned} \mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i &= -2\delta_{i,j} \mathbf{e}_0, \quad i, j = 1, 2, 3 \\ \mathbf{e}_0 \mathbf{e}_i &= \mathbf{e}_i \mathbf{e}_0 = \mathbf{e}_i, \quad i = 0, 1, 2, 3. \end{aligned}$$

This non-commutative product generates the algebra of real quaternions denoted by \mathbb{H} . The real vector space \mathbb{R}^4 will be embedded in \mathbb{H} by identifying $\mathbf{a} := (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with the element

$$\mathbf{a} = a_0 \mathbf{e}_0 + a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \in \mathbb{H},$$

where a_i ($i = 0, 1, 2, 3$) are real numbers. Remark that $\mathbf{e}_0 = (1, 0, 0, 0)^T$ is the multiplicative unit element of \mathbb{H} and by identifying \mathbf{e}_0 with 1, it will therefore neglected in the following notation.

The real number $\mathbf{Sca} := a_0$ is called the scalar part of \mathbf{a} and $\mathbf{Veca} := a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$ is the vector part of \mathbf{a} . Analogously to the complex case, the conjugate of \mathbf{a} is the quaternion $\bar{\mathbf{a}} := a_0 - a_1 \mathbf{e}_1 - a_2 \mathbf{e}_2 - a_3 \mathbf{e}_3$. The norm of \mathbf{a} is given by $|\mathbf{a}| = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^{1/2}$ and coincides with the corresponding Euclidean norm of \mathbf{a} , as a vector in \mathbb{R}^4 . Considering the subset

$$\mathcal{A} := \text{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2\}$$

of \mathbb{H} , the real vector space \mathbb{R}^3 can be embedded in \mathcal{A} by the identification of each element $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion

$$\mathbf{x} = x_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 \in \mathcal{A}.$$

As a consequence, we will often use the same symbol \mathbf{x} to represent a point in \mathbb{R}^3 as well as to represent the corresponding reduced quaternion. Note that the set \mathcal{A} is only a real vector space but not a sub-algebra of \mathbb{H} .

Let us consider an open set $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary. An \mathbb{H} -valued function is a mapping $f : \Omega \longrightarrow \mathbb{H}$ such that

$$f(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x})\mathbf{e}_1 + f_2(\mathbf{x})\mathbf{e}_2 + f_3(\mathbf{x})\mathbf{e}_3,$$

where the coordinates f_i are real-valued functions defined in Ω . For continuously real-differentiable functions $f : \Omega \longrightarrow \mathbb{H}$, the operator

$$D = \partial_{x_0} + \mathbf{e}_1 \partial_{x_1} + \mathbf{e}_2 \partial_{x_2} \tag{1}$$

is called the generalized Cauchy-Riemann operator. We define the conjugate generalized Cauchy-Riemann operator by

$$\overline{D} = \partial_{x_0} - \mathbf{e}_1 \partial_{x_1} - \mathbf{e}_2 \partial_{x_2}. \quad (2)$$

A function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{H}$ is called *left* (resp. *right*) *monogenic* in Ω if

$$Df = 0 \text{ in } \Omega \text{ (resp., } fD = 0 \text{ in } \Omega).$$

From now on we only use left monogenic functions. For simplicity, we will call them monogenic. The generalized Cauchy-Riemann operator (1) and its conjugate (2) factorize the Laplace operator in \mathbb{R}^3 . In fact, it holds

$$\Delta_3 = D\overline{D} = \overline{D}D$$

and implies that any monogenic function is also a harmonic function.

From now on, we will consider the following notations: $B := B_1(0)$ is the unit ball in \mathbb{R}^3 centered at the origin, $S = \partial B$ its boundary and $d\sigma$ is the Lebesgue measure on S . In what follows, we will denote by $L_2(S; \mathbb{X}; \mathbb{F})$ (resp. $L_2(B; \mathbb{X}; \mathbb{F})$) the \mathbb{F} -linear Hilbert space of square integrable functions on S (resp. B) with values in \mathbb{X} ($\mathbb{X} = \mathbb{R}$ or \mathcal{A} or \mathbb{H}), where $\mathbb{F} = \mathbb{H}$ or \mathbb{R} . For any $f, g \in L_2(S; \mathcal{A}; \mathbb{R})$ the real-valued inner product is given by

$$\langle f, g \rangle_{L_2(S)} = \int_S \mathbf{Sc}(\overline{f}g) d\sigma. \quad (3)$$

Each homogeneous harmonic polynomial P_n of order n can be written in spherical coordinates as

$$P_n(x) = r^n P_n(\omega), \quad \omega \in S, \quad (4)$$

its restriction, $P_n(\omega)$, to the boundary of the unit ball is called *spherical harmonic* of degree n . From (4), it is clear that a homogeneous polynomial is determined by its restriction to S . Denoting by $\mathcal{H}_n(S)$ the space of real-valued spherical harmonics of degree n in S , it is well-known (see [3] and [16]) that

$$\dim \mathcal{H}_n(S) = 2n + 1.$$

It is also known (see [3] and [16]) that if $n \neq m$, the spaces $\mathcal{H}_n(S)$ and $\mathcal{H}_m(S)$ are orthogonal in $L_2(S; \mathbb{R}; \mathbb{R})$.

Homogeneous monogenic polynomial of degree n will be denoted in general by H_n . In an analogous way to the spherical harmonics, the restriction of H_n to the boundary of the unit ball is called *spherical monogenic* of degree n . We denote by $\mathcal{M}_n(\mathbb{H}; \mathbb{F})$ the subspace of $L_2(B; \mathbb{H}; \mathbb{F}) \cap \ker D(B)$ of all homogeneous monogenic polynomials of degree n . Sudbery proved in [17] that the dimension of $\mathcal{M}_n(\mathbb{H}; \mathbb{H})$ is $n + 1$. In [5], it is proved that the dimension of $\mathcal{M}_n(\mathbb{H}; \mathbb{R})$ is $4n + 4$.

Consider, for each $n \in \mathbb{N}_0$, a basis $\{H_n^\nu : \nu = 1, \dots, \dim \mathcal{M}_n(\mathbb{H}; \mathbb{F})\}$ of $\mathcal{M}_n(\mathbb{H}; \mathbb{F})$, $\mathbb{F} = \mathbb{H}$ or $\mathbb{F} = \mathbb{R}$. Taking into account that the coordinates of H_n^ν are harmonic, for arbitrary $n, k = 0, 1, \dots$, we have

$$\langle H_n^\nu, H_k^\mu \rangle_{L_2(B; \mathbb{H}; \mathbb{F})} = \delta_{n,k} \frac{1}{n+k+3} \langle H_n^\nu, H_k^\mu \rangle_{L_2(S; \mathbb{H}; \mathbb{F})}. \quad (5)$$

3 Homogeneous Monogenic Polynomials

Based on the Fueter variables $\mathbf{z}_1 = x_1 - \mathbf{e}_1 x_0$ and $\mathbf{z}_2 = x_2 - \mathbf{e}_2 x_0$, several systems of homogeneous monogenic polynomials are constructed and used for different purposes (see, e.g., [4, 9, 7, 10, 12, 17]). Following [12], being $\underline{\gamma} = (\gamma_1, \gamma_2)$ a multi-index with $\gamma_1 + \gamma_2 = n$, the generalized powers (or also Fueter polynomials) of degree n are defined by

$$\begin{aligned} \mathbf{z}_1^{\gamma_1} \times \mathbf{z}_2^{\gamma_2} &= \underbrace{\mathbf{z}_1 \times \mathbf{z}_1 \times \cdots \times \mathbf{z}_1}_{\gamma_1 \text{ times}} \times \underbrace{\mathbf{z}_2 \times \mathbf{z}_2 \times \cdots \times \mathbf{z}_2}_{\gamma_2 \text{ times}} \\ &= \frac{1}{n!} \sum_{\pi(i_1, \dots, i_n)} \mathbf{z}_{i_1} \cdots \mathbf{z}_{i_n}, \end{aligned}$$

where the sum is taken over all permutations $\pi(i_1, \dots, i_n)$ of $(\underbrace{1, \dots, 1}_{\gamma_1}, \underbrace{2, \dots, 2}_{\gamma_2})$.

The general form of the Taylor series of a monogenic function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{H}$ in the neighborhood of the origin (see, e.g., [4, 12]) is given by

$$f = \sum_{n=0}^{\infty} \sum_{|\underline{\gamma}|=n} (\mathbf{z}_1^{\gamma_1} \times \mathbf{z}_2^{\gamma_2}) c_{\underline{\gamma}}, \quad (6)$$

where $c_{\underline{\gamma}} = \frac{1}{\gamma_1! \gamma_2!} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} f(\mathbf{x}) \Big|_{\mathbf{x}=0} \in \mathbb{H}$ are the Taylor coefficients.

In order to prove a collection of inequalities related to Bohr's inequality, we need also the Fourier expansion of monogenic functions.

In ([5] and [6]) \mathbb{R} -linear and \mathbb{H} -linear complete orthonormal systems of \mathbb{H} -valued homogeneous monogenic polynomials in the unit ball of \mathbb{R}^3 are constructed. The main idea of these constructions is based on the factorization of the Laplace operator. We take a system of real-valued homogeneous harmonic polynomials and apply the \overline{D} operator to get systems of \mathbb{H} -valued homogeneous monogenic polynomials. To be precise, we introduce the spherical coordinates,

$$x_0 = r \cos \theta, \quad x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi,$$

where $0 < r < \infty$, $0 < \theta \leq \pi$, $0 < \varphi \leq 2\pi$. Each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}$ admits a unique representation $\mathbf{x} = r\mathbf{w}$, where for each $i = 0, 1, 2$ $w_i = \frac{x_i}{r}$ and $|\mathbf{w}| = 1$. Now, we apply for each $n \in \mathbb{N}_0$, the operator $\frac{1}{2}\overline{D}$ to the homogeneous harmonic polynomials,

$$\{r^{n+1}U_{n+1}^0, r^{n+1}U_{n+1}^m, r^{n+1}V_{n+1}^m, m = 1, \dots, n+1\}_{n \in \mathbb{N}_0} \quad (7)$$

formed by the extensions in the ball of the spherical harmonics (considered, e.g., in [18]),

$$\begin{aligned} U_{n+1}^0(\theta, \varphi) &= P_{n+1}(\cos \theta) \\ U_{n+1}^m(\theta, \varphi) &= P_{n+1}^m(\cos \theta) \cos m\varphi \\ V_{n+1}^m(\theta, \varphi) &= P_{n+1}^m(\cos \theta) \sin m\varphi, m = 1, \dots, n+1. \end{aligned} \quad (8)$$

Here, P_{n+1} stands for the Legendre polynomial of degree $n + 1$, given by

$$\begin{cases} P_{n+1}(t) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} a_{n+1,k} t^{n+1-2k} \\ P_0(t) = 1, \quad t \in (-1, 1), \end{cases}$$

with

$$a_{n+1,k} = (-1)^k \frac{1}{2^{n+1}} \frac{(2n+2-2k)!}{k!(n+1-k)!(n+1-2k)!},$$

where $[s]$ denotes the integer part of $s \in \mathbb{R}$. Also, we stipulate this sum to be zero whenever the upper index is less than the lower one.

The functions P_{n+1}^m are called the associated Legendre functions,

$$P_{n+1}^m(t) := (1-t^2)^{m/2} \frac{d^m}{dt^m} P_{n+1}(t), \quad m = 1, \dots, n+1. \quad (9)$$

We remark that if $m = 0$, the corresponding associated Legendre function P_{n+1}^0 coincides with the Legendre polynomial P_{n+1} .

Notice that the Legendre polynomials together with the associated Legendre functions satisfy several recurrence formulae. We point out only some of them, which will be used several times in the next section. Following [2], Legendre polynomials and associated Legendre functions are solutions of a second order differential equation, called *Legendre differential equation*, given by

$$(1-t^2)(P_{n+1}^m(t))'' - 2t(P_{n+1}^m(t))' + \left((n+1)(n+2) - m^2 \frac{1}{1-t^2} \right) P_{n+1}^m(t) = 0,$$

$m = 0, \dots, n+1$. They also satisfy the recurrence formula

$$(1-t^2)(P_{n+1}^m(t))' = (n+m+1)P_n^m(t) - (n+1)tP_{n+1}^m(t), \quad (10)$$

$m = 0, \dots, n+1$. An additional and useful identity is given by

$$P_m^m(t) = (2m-1)!!(1-t^2)^{m/2}, \quad (11)$$

$m = 1, \dots, n+1$.

These functions are mutually orthogonal in $L_2([-1, 1])$,

$$\int_{-1}^1 P_{n+1}^m(t) P_{k+1}^m(t) dt = 0, \quad n \neq k$$

and their norms are

$$\int_{-1}^1 (P_{n+1}^m(t))^2 dt = \frac{2}{2n+3} \frac{(n+1+m)!}{(n+1-m)!}, \quad m = 0, \dots, n+1.$$

For a detailed study of Legendre polynomials and associated Legendre functions we refer, for example, [2] and [18].

Restricting the functions of the set (7) to the sphere, we obtain the spherical monogenics

$$\begin{aligned} X_n^0 &:= \left(\frac{1}{2} \overline{D} \right) (r^{n+1} U_{n+1}^0) \Big|_{r=1} \\ X_n^m &:= \left(\frac{1}{2} \overline{D} \right) (r^{n+1} U_{n+1}^m) \Big|_{r=1} \\ Y_n^m &:= \left(\frac{1}{2} \overline{D} \right) (r^{n+1} V_{n+1}^m) \Big|_{r=1}, \quad m = 1, \dots, n+1. \end{aligned} \quad (12)$$

For each $n \in \mathbb{N}_0$, taking the monogenic extensions of the spherical monogenics into the ball, we obtain the set of homogeneous monogenic polynomials

$$\{r^n X_n^0, r^n X_n^m, r^n Y_n^m : m = 1, \dots, n+1\}. \quad (13)$$

We need norm estimates of our functions in terms of its Taylor and Fourier expansion are needed. In this way, we begin now to write the homogeneous monogenic polynomials in Cartesian coordinates. In parts, these results were already obtained in [13] and [14], without proof.

Lemma 3.1 *The homogeneous monogenic polynomials $r^n X_n^l$ ($l = 0, 1, \dots, n+1$) in terms of Cartesian coordinates can be written as:*

$$r^n X_n^l(\mathbf{x}) = [r^n X_n^l(\mathbf{x})]_0 + [r^n X_n^l(\mathbf{x})]_1 \mathbf{e}_1 + [r^n X_n^l(\mathbf{x})]_2 \mathbf{e}_2,$$

where

$$\begin{aligned} [r^n X_n^l(\mathbf{x})]_0 &= \sum_{k=0}^{\lfloor \frac{n-l}{2} \rfloor} \beta_{n+1,l,k} (n+1-2k-l) x_0^{n-2k-l} r^{2k} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^j \binom{l}{2j} x_1^{l-2j} x_2^{2j} \\ &+ \sum_{k=1}^{\lfloor \frac{n+1-l}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+2-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^j \binom{l}{2j} x_1^{l-2j} x_2^{2j} \\ [r^n X_n^l(\mathbf{x})]_1 &= \sum_{k=1}^{\lfloor \frac{n+1-l}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+1-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} x_1^{l-2j+1} x_2^{2j} \\ &+ \sum_{k=0}^{\lfloor \frac{n+1-l}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} \sum_{j=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} (l-2j) x_1^{l-2j-1} x_2^{2j} \\ [r^n X_n^l(\mathbf{x})]_2 &= \sum_{k=1}^{\lfloor \frac{n+1-l}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+1-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} x_1^{l-2j} x_2^{2j+1} \\ &+ \sum_{k=0}^{\lfloor \frac{n+1-l}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} (2j) x_1^{l-2j} x_2^{2j-1}, \end{aligned}$$

being

$$\beta_{n+1,l,k} = (-1)^k \frac{1}{2^{n+2}} \binom{2n+2-2k}{n+1-k} \binom{n+1-k}{k} (n+1-2k)_{l-1}$$

and $(n+1-2k)_{l-1}$ stands for the Pochhammer symbol.

Proof. Let us consider the spherical monogenics given by (12), explicitly described in (8). By the definition of the Legendre polynomials we have

$$P_{n+1}^{(1)}(t) = \frac{d}{dt} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} a_{n+1,k} t^{n+1-2k} = \sum_{k=0}^{\lfloor \frac{(n+1)-1}{2} \rfloor} a_{n+1,k}(n+1-2k) t^{(n+1-2k)-1}.$$

Now, derivating recursively in order to $t^{(l-1)}$ times,

$$\begin{aligned} \partial_t^l P_{n+1}(t) &= P_{n+1}^{(l)}(t) \\ &= \sum_{k=0}^{\lfloor \frac{(n+1)-l}{2} \rfloor} a_{n+1,k}(n+1-2k)(n+1-2k-1) \cdots (n+1-2k-(l-1)) t^{(n+1-2k)-l}. \end{aligned}$$

By simplicity, we set

$$\beta_{n+1,l,k} = 2(n+1-2k)(n+1-2k-1) \cdots (n+1-2k-(l-1)),$$

so that, finally we get for (9) the expression

$$P_{n+1}^l(\cos \theta) = \sum_{k=0}^{\lfloor \frac{n+1-l}{2} \rfloor} 2\beta_{n+1,l,k} (\sin \theta)^l (\cos \theta)^{n+1-2k-l}.$$

In order to express the set $\{X_n^l : l = 0, 1, \dots, n+1\}$ in cartesian coordinates, we consider the coordinate's relation:

$$\begin{aligned} \cos \theta &= \frac{x_0}{r} & \cos \varphi &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ \sin \theta &= \frac{\sqrt{x_1^2 + x_2^2}}{r} & \sin \varphi &= \frac{x_2}{\sqrt{x_1^2 + x_2^2}}. \end{aligned}$$

Now, using

$$\cos(m\varphi) = \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^j \binom{l}{2j} (\cos \varphi)^{l-2j} (\sin \varphi)^{2j}$$

and substituting in (12) we obtain

$$r^{n+1} U_{n+1}^l(\mathbf{x}) = 2 \sum_{k=0}^{\lfloor \frac{n+1-l}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} (-1)^j \binom{l}{2j} x_1^{l-2j} x_2^{2j}.$$

Applying the hypercomplex derivative $(\frac{1}{2}\overline{D})$ to this expression carries our polynomials in Cartesian coordinates, respectively. ■

Similar results holds for $r^n Y_n^m$ $m = 1, \dots, n+1$). Let us consider now the following function:

Definition 3.1 Let $i, j \in \mathbb{N}_0$. The function $g_{i,j}$ is given by

$$g_{i,j} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ have the same parity} \\ 0, & \text{if } i \text{ and } j \text{ have different parity} \end{cases}.$$

Proposition 3.1 *The Taylor coefficients of the homogeneous monogenic polynomials $r^n X_n^l$ ($l = 0, 1, \dots, n+1$) are given by*

$$\begin{aligned}
[a_{\underline{\alpha}}^l]_0 &= g_{l,n} g_{\alpha_1,l} g_{\alpha_2,0} \beta_{n+1,l,\frac{n-l}{2}} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^j \binom{l}{2j} \binom{\frac{n-l}{2}}{\frac{\alpha_1-l}{2}+j} \\
[a_{\underline{\alpha}}^l]_1 &= g_{l-1,n} g_{l-1,\alpha_1} g_{\alpha_2,0} \\
&\quad \left[\sum_{p=1}^{\lfloor \frac{n-l+1}{2} \rfloor} \beta_{n+1,l,p}(2p) \binom{p-1}{\frac{l-n-1}{2}+p} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} \binom{\frac{n-l-1}{2}}{\frac{\alpha_1-l}{2}+j} \right. \\
&\quad \left. + \sum_{p=0}^{\lfloor \frac{n-l+1}{2} \rfloor} \beta_{n+1,l,p} \binom{p}{\frac{l-n-1}{2}+p} \sum_{j=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} (l-2j) \binom{\frac{n-l+1}{2}}{\frac{\alpha_1-l}{2}+j} \right] \\
[a_{\underline{\alpha}}^l]_2 &= g_{l-1,n} g_{l,\alpha_1} g_{\alpha_2,1} \\
&\quad \left[\sum_{p=1}^{\lfloor \frac{n-l+1}{2} \rfloor} \beta_{n+1,l,p}(2p) \binom{p-1}{\frac{l-n-1}{2}+p} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} \binom{\frac{n-l-1}{2}}{\frac{\alpha_1-l}{2}+j} \right. \\
&\quad \left. + \sum_{p=0}^{\lfloor \frac{n-l+1}{2} \rfloor} \beta_{n+1,l,p} \binom{p}{\frac{l-n-1}{2}+p} \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} (-1)^{j+1} \binom{l}{2j} (2j) \binom{\frac{n-l+1}{2}}{\frac{\alpha_1-l}{2}+j} \right]
\end{aligned}$$

Proof. The proof follows directly from Lemma 3.1 by applying the partial derivatives $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$. ■

Again, we obtain analogous formulae for the Taylor coefficients of $r^n Y_n^m$ ($m = 1, \dots, n+1$).

Corollary 3.1 *Let $\underline{\gamma} = (\gamma_1, \gamma_2)$ be a multi-index with $|\underline{\gamma}| = n$. The Taylor coefficients $a_{\underline{\gamma}}^0$, $a_{\underline{\gamma}}^m$ and $b_{\underline{\gamma}}^m$ of the homogeneous monogenic polynomials $r^n X_n^0$, $r^n X_n^m$ and $r^n Y_n^m$ satisfy the following inequalities:*

$$\begin{aligned}
|a_{\underline{\gamma}}^0| &\leq \frac{1}{\underline{\gamma}!} (n+1)! \sqrt{\frac{\pi(n+1)}{2n+3}} \\
|a_{\underline{\gamma}}^m| &\leq \frac{1}{\underline{\gamma}!} (n+1)! \sqrt{\frac{\pi(n+1)(n+1+m)!}{2(2n+3)(n+1-m)!}} \\
|b_{\underline{\gamma}}^m| &\leq \frac{1}{\underline{\gamma}!} (n+1)! \sqrt{\frac{\pi(n+1)(n+1+m)!}{2(2n+3)(n+1-m)!}}, \quad m = 1, \dots, n+1.
\end{aligned}$$

Proof. Let $B_r(\mathbf{x}) \subset \mathbb{R}^3$ be a ball of radius r centered at \mathbf{x} . From [11] we know the Cauchy integral formula for the ball $B_1(\mathbf{x})$,

$$f(\mathbf{x}) = \frac{1}{4\pi} \int_S \frac{\overline{\mathbf{x} - \mathbf{y}}}{|\mathbf{x} - \mathbf{y}|^3} \mathbf{n}(\mathbf{y}) f(\mathbf{y}) dS_{\mathbf{y}}, \quad (14)$$

where \mathbf{n} stands for the outward pointing normal unit vector to S at \mathbf{y} . For simplicity we just present the proof for the homogeneous monogenic polynomials $r^n X_n^m$ ($m = 1, \dots, n+1$). Applying the Cauchy integral formula to these polynomials in the ball B and taking partial derivatives with respect to x_1 and x_2 , we get

$$a_{\underline{\gamma}}^{m,*} = \frac{1}{\underline{\gamma}!} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} X_n^m(\mathbf{x})|_{\mathbf{x}=0} = \frac{1}{\underline{\gamma}!} \frac{1}{4\pi} \int_S \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \frac{\overline{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3} \Big|_{\mathbf{x}=0} \mathbf{n}(\mathbf{y}) X_n^m(\mathbf{y}) dS_{\mathbf{y}}$$

taking the modulus and applying the Schwarz inequality we finally obtain

$$|a_{\underline{\gamma}}^{m,*}| \leq \frac{1}{\underline{\gamma}!} (n+1)! \sqrt{\frac{\pi}{2} (n+1) \frac{(n+1+m)!}{(n+1-m)!}},$$

where $a_{\underline{\gamma}}^{m,*}$ denotes the Taylor coefficients associated to the functions X_n^m . The previous inequality is based on [5] where the following relation is proved

$$\|X_n^m\|_{L_2(S)} = \|Y_n^m\|_{L_2(S)} = \sqrt{\frac{\pi}{2} (n+1) \frac{(n+1+m)!}{(n+1-m)!}}, \quad m = 1, \dots, n+1$$

and on the paper [26] where it was obtained that

$$\left\| \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \frac{\overline{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3} \Big|_{\mathbf{x}=0} \right\|_{L_2(S)} \leq \frac{(n+1)!}{|\mathbf{y}|^{n+2}}.$$

Using the relation (5) we get the Taylor coefficients associated to the homogeneous monogenic polynomials $r^n X_n^m$. The case $m = 0$ is proved analogously. ■

Besides norm estimates we also need pointwise estimates of our basis polynomials.

Proposition 3.2 *For $n \in \mathbb{N}$ the homogeneous monogenic polynomials satisfy the following inequalities:*

$$\begin{aligned} |r^n X_n^0(\mathbf{x})| &\leq r^n (n+1) 2^n \sqrt{\frac{\pi(n+1)}{2n+3}} \\ |r^n X_n^m(\mathbf{x})| &\leq r^n (n+1) 2^n \sqrt{\frac{\pi}{2} \frac{(n+1)}{(2n+3)} \frac{(n+1+m)!}{(n+1-m)!}} \\ |r^n Y_n^m(\mathbf{x})| &\leq r^n (n+1) 2^n \sqrt{\frac{\pi}{2} \frac{(n+1)}{(2n+3)} \frac{(n+1+m)!}{(n+1-m)!}}, \quad m = 1, \dots, n+1. \end{aligned}$$

Proof. Again, we prove only the case of the polynomials $r^n X_n^m$ ($m = 1, \dots, n+1$), the proof for $r^n Y_n^m$ being similar. We write these polynomials as a Taylor expansion (6)

$$r^n X_n^m(\mathbf{x}) = \sum_{|\underline{\gamma}|=n} (\mathbf{z}_1^{\gamma_1} \times \mathbf{z}_2^{\gamma_2}) a_{\underline{\gamma}}^m.$$

Consequently, modulus of $r^n X_n^m$ leads to

$$|r^n X_n^m(\mathbf{x})| \leq r^n (n+1)! \sqrt{\frac{\pi}{2} \frac{(n+1)}{(2n+3)} \frac{(n+1+m)!}{(n+1-m)!} \frac{2^n}{n!}}.$$

Having in mind [12] we have

$$|\mathbf{z}_1^{\gamma_1} \times \mathbf{z}_2^{\gamma_2}| \leq r^n$$

for every multi-index $\underline{\gamma} = (\gamma_1, \gamma_2)$ with $|\underline{\gamma}| = n$.

For future use in this paper we will need estimates for the real part of the spherical monogenics described in (12).

Theorem 3.1 *Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\{\mathbf{Sc}\{X_n^0\}, \mathbf{Sc}\{X_n^m\}, \mathbf{Sc}\{Y_n^m\} : m = 1, \dots, n\}$$

are orthogonal to each other with respect to the inner product (3).

Proposition 3.3 *Given a fixed $n \in \mathbb{N}_0$, the moduli of the spherical harmonics $\mathbf{Sc}(X_n^0)$, $\mathbf{Sc}(X_n^m)$ and $\mathbf{Sc}(Y_n^m)$ satisfy the following inequalities*

$$\begin{aligned} |\mathbf{Sc}\{X_n^l\}| &\leq \frac{1}{2} \frac{(n+1+l)!}{n!}, \quad l = 0, \dots, n \\ |\mathbf{Sc}\{Y_n^m\}| &\leq \frac{1}{2} \frac{(n+1+m)!}{n!}, \quad m = 1, \dots, n. \end{aligned}$$

Proof. According to the results from [5], the real parts of the spherical monogenics are given by

$$\begin{aligned} \mathbf{Sc}\{X_n^0\} &= A^{0,n}(\theta) \\ \mathbf{Sc}\{X_n^m\} &= A^{m,n}(\theta) \cos(m\varphi) \\ \mathbf{Sc}\{Y_n^m\} &= A^{m,n}(\theta) \sin(m\varphi), \end{aligned}$$

where

$$A^{l,n}(\theta) = \frac{1}{2} \left(\sin^2 \theta \frac{d}{dt} [P_{n+1}^l(t)]_{t=\cos \theta} + (n+1) \cos \theta P_{n+1}^l(\cos \theta) \right), \quad l = 0, \dots, n.$$

For simplicity sake we only present the proof for the spherical harmonics $\mathbf{Sc}(X_n^m)$ ($m = 1, \dots, n+1$). Making the change of variable $t = \cos \theta$ and using the recurrence formula (10), it follows that

$$\mathbf{Sc}\{X_n^m\} = \frac{1}{2} (n+1+m) P_n^m(t).$$

Applying the modulus in the previous expression and using the inequality proved in [19]

$$|P_n^m(t)| \leq \frac{(n+m)!}{n!},$$

for $-1 \leq t \leq 1$ and $n \geq m$, we finally obtain the estimate

$$|\mathbf{Sc}\{X_n^m\}| \leq \frac{1}{2} \frac{(n+1+m)!}{n!}. \blacksquare$$

Some of the basis polynomials described in (13) play a special role. Applying results from ([5], Proposition 3.4.3) we get:

Proposition 3.4 For $n \in \mathbb{N}_0$, the spherical monogenics X_n^{n+1} and Y_n^{n+1} are given by

$$\begin{aligned} X_n^{n+1} &= -C^{n+1,n} \cos n\varphi \mathbf{e}_1 + C^{n+1,n} \sin n\varphi \mathbf{e}_2 \\ Y_n^{n+1} &= -C^{n+1,n} \sin n\varphi \mathbf{e}_1 - C^{n+1,n} \cos n\varphi \mathbf{e}_2 \end{aligned} \quad (15)$$

where

$$C^{n+1,n} = \frac{n+1}{2} \frac{1}{\sin \theta} P_{n+1}^{n+1}(\cos \theta),$$

and their monogenic extensions into the ball belong to $\ker \overline{D}(B) \cap \ker D(B)$.

Remark 3.1 The spherical monogenics X_n^{n+1} and Y_n^{n+1} are monogenic constants, i.e., monogenic functions which depend only on x_1 and x_2 . Moreover, they play the role of constants with respect to the hypercomplex differentiation $(\frac{1}{2}\overline{D})$.

Proposition 3.5 Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics $\mathbf{Sc}(X_n^{n+1}\mathbf{e}_1)$ and $\mathbf{Sc}(Y_n^{n+1}\mathbf{e}_1)$ are orthogonal to each other with respect to the inner product (3) and their moduli satisfy the following inequalities

$$\begin{aligned} |\mathbf{Sc}\{X_n^{n+1}\mathbf{e}_1\}| &\leq \frac{1}{2} \frac{(n+1)(2n+1)!}{2^n n!} \\ |\mathbf{Sc}\{Y_n^{n+1}\mathbf{e}_1\}| &\leq \frac{1}{2} \frac{(n+1)(2n+1)!}{2^n n!}. \end{aligned}$$

Proof. Again, we present the proof for the spherical harmonics $\mathbf{Sc}\{X_n^{n+1}\mathbf{e}_1\}$, the one for $\mathbf{Sc}\{Y_n^{n+1}\mathbf{e}_1\}$ being similar. According to (15), the real part of the spherical harmonic $X_n^{n+1}\mathbf{e}_1$ is given by

$$\mathbf{Sc}\{X_n^{n+1}\mathbf{e}_1\} = C^{n+1,n} \cos n\varphi.$$

Making the change of variable $t = \cos \theta$ and applying the modulus in the previous expression, we get

$$|\mathbf{Sc}\{X_n^{n+1}\mathbf{e}_1\}| = \frac{n+1}{2} \left| \frac{1}{\sqrt{1-t^2}} P_{n+1}^{n+1}(t) \right|,$$

and due to the recurrence formula (11) we finally obtain

$$|\mathbf{Sc}\{X_n^{n+1}\mathbf{e}_1\}| = \frac{n+1}{2} \left| \frac{1}{\sqrt{1-t^2}} (2n+1)!! (1-t^2)^{\frac{n+1}{2}} \right| \leq \frac{1}{2} (n+1)(2n+1)!! \quad \blacksquare$$

Proposition 3.6 Given a fixed $n \in \mathbb{N}_0$, the norms of the spherical harmonics $\mathbf{Sc}(X_n^0)$, $\mathbf{Sc}(X_n^m)$ and $\mathbf{Sc}(Y_n^m)$ are given by

$$\|\mathbf{Sc}(X_n^0)\|_{L_2(S)} = (n+1) \sqrt{\frac{\pi}{2n+1}}$$

and

$$\|\mathbf{Sc}(X_n^m)\|_{L_2(S)} = \|\mathbf{Sc}(Y_n^m)\|_{L_2(S)} = \sqrt{\frac{\pi}{2} \frac{(n+1+m)(n+1+m)!}{(2n+1)(n-m)!}}, \quad m = 1, \dots, n.$$

Proposition 3.7 *Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics $\mathbf{Sc}(X_n^{n+1}\mathbf{e}_1)$ and $\mathbf{Sc}(Y_n^{n+1}\mathbf{e}_1)$ are orthogonal to each other with respect to the inner product (3) and their norms are given by*

$$\|\mathbf{Sc}(X_n^{n+1}\mathbf{e}_1)\|_{L_2(S)} = \|\mathbf{Sc}(Y_n^{n+1}\mathbf{e}_1)\|_{L_2(S)} = \frac{1}{2}\sqrt{\pi(n+1)(2n+2)!}.$$

4 Bohr's Theorem

We will denote by $X_n^{0,*}, \dots$ the normalized basis functions in $L_2(S; \mathbb{H}; \mathbb{H})$.

Theorem 4.1 (see [5]) *Let $M_n(\mathbb{R}^3; \mathcal{A})$ be the space of \mathcal{A} -valued homogeneous monogenic polynomials of degree n in \mathbb{R}^3 . For each n , the set of $2n+3$ homogeneous monogenic polynomials*

$$\{\sqrt{2n+3}r^n X_n^{0,*}, \sqrt{2n+3}r^n X_n^{m,*}, \sqrt{2n+3}r^n Y_n^{m,*}, m = 1, \dots, n+1\} \quad (16)$$

forms an orthonormal basis in $M_n(\mathbb{R}^3; \mathcal{A})$.

In [15], a first version of a quaternionic Bohr's theorem was considered, therein we restricted ourselves to the case of functions with $f(0) = 0$ and we obtained an estimate in terms of a radius of $r = 0.047$.

Here, we extend our result to all monogenic functions with $|f(\mathbf{x})| < 1$ in B , estimating a value for the radius.

Theorem 4.2 *Let f be a square integrable \mathcal{A} -valued monogenic function with $|f(\mathbf{x})| < 1$ in B , $\mathbf{Sc}\{f\}$ be positive and let*

$$\sum_{n=0}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\}$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty} \sqrt{2n+3} r^n \left| \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\} \right| < 1$$

holds in the ball of radius r , with $0 \leq r < 0.05$.

Proof. According to Theorem 4.1, a monogenic L_2 -function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathcal{A}$ can be written as Fourier series

$$f = \sum_{n=0}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\},$$

where α_n^0, α_n^m and β_n^m ($m = 1, \dots, n+1$) are the associated Fourier coefficients. Let us denote by $\mathbf{Sc}\{f\}$ the real part of f . Then,

$$\begin{aligned} \mathbf{Sc}\{f\} &= \frac{f + \bar{f}}{2} \\ &= \sum_{n=0}^{\infty} \sqrt{2n+3} r^n \left\{ \mathbf{Sc}\{X_n^{0,*}\} \alpha_n^0 + \sum_{m=1}^n [\mathbf{Sc}\{X_n^{m,*}\} \alpha_n^m + \mathbf{Sc}\{Y_n^{m,*}\} \beta_n^m] \right\}. \end{aligned}$$

Due to Remark 3.1, we split the function f in the following way

$$\begin{aligned} f &= \sqrt{3}\alpha_0^0 X_0^{0,*} + \sqrt{3}\alpha_0^1 X_0^{1,*} + \sqrt{3}\beta_0^1 Y_0^{1,*} \\ &+ \sum_{n=1}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^n [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\} \\ &+ \sum_{n=1}^{\infty} \sqrt{2n+3} r^n [X_n^{n+1,*} \alpha_n^{n+1} + Y_n^{n+1,*} \beta_n^{n+1}]. \end{aligned}$$

Based in this splitting, we introduce

$$\begin{aligned} f_1 &= \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^n [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\} \\ f_2 &= -\frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \mathbf{e}_1 - \frac{1}{2} \sqrt{\frac{3}{\pi}} \beta_0^1 \mathbf{e}_2 + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n [X_n^{n+1,*} \alpha_n^{n+1} + Y_n^{n+1,*} \beta_n^{n+1}], \end{aligned}$$

so that $f = f_1 + f_2$. Then, we have

$$f(0) = f_1(0) + f_2(0)$$

where

$$\begin{aligned} f_1(0) &= \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \\ f_2(0) &= -\frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \mathbf{e}_1 - \frac{1}{2} \sqrt{\frac{3}{\pi}} \beta_0^1 \mathbf{e}_2. \end{aligned}$$

Let us assume that there exists $0 < \delta < 1$ such that $|f_1| < \delta$ and $|f_2| < 1 - \delta$. In this way, the modulus of f is preserved. We start now to study the function f_1 . The main idea is to compare each Fourier coefficient with the coefficient α_0^0 . In fact, multiplying both sides of the expression

$$\mathbf{Sc}\{\delta - f_1\} = \delta - \mathbf{Sc}\{f_1\} \quad (17)$$

by each real part of the homogeneous monogenic polynomials described in (13) and integrating over the sphere, we get these relations. For simplicity we just present the idea applied to the coefficients of $X_n^{0,*}$, i.e, α_n^0 . Multiplying both sides of the expression (17) by $\mathbf{Sc}\{X_k^0\}$ and integrating, we obtain

$$-\sqrt{2k+3} \alpha_k^0 = \int_S \mathbf{Sc}\{\delta - f_1\} \mathbf{Sc}\{X_k^0\} d\sigma$$

with $0 < \delta < 1$. Now, applying the modulus we obtain finally

$$|\alpha_k^0| \sqrt{2k+3} \leq 2\sqrt{\pi} \frac{|\mathbf{Sc}\{X_k^0\}|}{\|\mathbf{Sc}\{X_k^0\}\|_{L_2(S)}^2} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right). \quad (18)$$

In an analogous way, we can state the following results:

$$\begin{aligned} |\alpha_k^p| \sqrt{2k+3} &\leq 2\sqrt{\pi} \frac{|\mathbf{Sc}\{X_k^p\}|}{\|\mathbf{Sc}\{X_k^p\}\|_{L_2(S)}^2} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right), \\ |\beta_k^p| \sqrt{2k+3} &\leq 2\sqrt{\pi} \frac{|\mathbf{Sc}\{Y_k^p\}|}{\|\mathbf{Sc}\{Y_k^p\}\|_{L_2(S)}^2} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right), \quad p = 1, \dots, k. \end{aligned}$$

With some calculations and, using the Propositions 3.6 and 3.3 we finally obtain

$$\begin{aligned} \frac{|\mathbf{Sc}\{X_k^0\}|}{\|\mathbf{Sc}\{X_k^0\}\|_{L_2(S)}^2} &\leq \frac{1}{2\pi} \frac{(2k+1)}{k+1} \\ \frac{|\mathbf{Sc}\{X_k^p\}|}{\|\mathbf{Sc}\{X_k^p\}\|_{L_2(S)}^2} &= \frac{|\mathbf{Sc}\{Y_k^p\}|}{\|\mathbf{Sc}\{Y_k^p\}\|_{L_2(S)}^2} \leq \frac{1}{\pi} \frac{(2k+1)(k-p)!}{(k+1+p)k!}, \quad p = 1, \dots, k. \end{aligned}$$

Finally, the previous expressions can be rewritten

$$\begin{aligned} |\alpha_k^0| \sqrt{2k+3} &\leq \frac{1}{\sqrt{\pi}} \frac{(2k+1)}{k+1} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \\ |\alpha_k^p| \sqrt{2k+3} &\leq \frac{2}{\sqrt{\pi}} \frac{(2k+1)(k-p)!}{(k+1+p)k!} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \\ |\beta_k^p| \sqrt{2k+3} &\leq \frac{2}{\sqrt{\pi}} \frac{(2k+1)(k-p)!}{(k+1+p)k!} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right). \end{aligned}$$

Consequently, we can state the following inequalities:

$$\begin{aligned} |X_k^{0,*}| |\alpha_k^0| \sqrt{2k+3} &\leq \frac{1}{\sqrt{\pi}} (2r)^k (2k+1) \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \\ \sum_{p=1}^k |X_k^{p,*}| |\alpha_k^p| \sqrt{2k+3} &\leq \frac{2}{\sqrt{\pi}} (2r)^k (2k+1) \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \\ \sum_{p=1}^k |Y_k^{p,*}| |\beta_k^p| \sqrt{2k+3} &\leq \frac{2}{\sqrt{\pi}} (2r)^k (2k+1) \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right). \end{aligned}$$

Now, using the previous inequalities we end with

$$\begin{aligned} |f_1| &\leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n \left[|X_n^{0,*}| |\alpha_n^0| + \sum_{m=1}^n (|X_n^{m,*}| |\alpha_n^m| + |Y_n^{m,*}| |\beta_n^m|) \right] \\ &\leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \frac{5}{\sqrt{\pi}} \left(\delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \sum_{n=1}^{\infty} (2r)^n (2n+1). \end{aligned}$$

Thus, we have that

$$|f_1| \leq \delta \implies \frac{5}{\sqrt{\pi}} \sum_{n=1}^{\infty} (2r)^n (2n+1) \leq 1,$$

and, the last series is convergent for $r < 0.05$. In the same way, we can study the function f_2 . Let

$$f_2 = \sqrt{3} \alpha_0^1 r^0 X_0^{1,*} + \sqrt{3} \beta_0^1 r^0 Y_0^{1,*} + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n [X_n^{n+1,*} \alpha_n^{n+1} + Y_n^{n+1,*} \beta_n^{n+1}].$$

Multiplying f_2 in the right side by \mathbf{e}_1 we get

$$\begin{aligned}\tilde{f}_2 &:= f_2 \mathbf{e}_1 \\ &= \sqrt{3} \alpha_0^1 (r^0 X_0^{1,*} \mathbf{e}_1) + \sqrt{3} \beta_0^1 (r^0 Y_0^{1,*} \mathbf{e}_1) \\ &+ \sum_{n=1}^{\infty} \sqrt{2n+3} r^n [(X_n^{n+1,*} \mathbf{e}_1) \alpha_n^{n+1} + (Y_n^{n+1,*} \mathbf{e}_1) \beta_n^{n+1}].\end{aligned}$$

We want to apply the same idea previously used for f_1 . Taking in consideration that f is an \mathcal{A} -valued function, we obtain an estimate for the coefficient α_0^1 . In a similar way, we obtain an estimate for β_0^1 if we multiply f_2 at right by \mathbf{e}_2 . This leads to the inequalities

$$\begin{aligned}|\alpha_k^{k+1}| \sqrt{2k+3} &\leq 2 \sqrt{\frac{\pi}{3}} \frac{|\mathbf{Sc}\{X_k^{k+1} \mathbf{e}_1\}|}{\|\mathbf{Sc}\{X_k^{k+1} \mathbf{e}_1\}\|_{L_2(S)}^2} \left((1-\delta) - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right) \\ |\beta_k^{k+1}| \sqrt{2k+3} &\leq 2 \sqrt{\frac{\pi}{3}} \frac{|\mathbf{Sc}\{Y_k^{k+1} \mathbf{e}_1\}|}{\|\mathbf{Sc}\{Y_k^{k+1} \mathbf{e}_1\}\|_{L_2(S)}^2} \left((1-\delta) - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right).\end{aligned}$$

being

$$\frac{|\mathbf{Sc}\{X_k^{k+1} \mathbf{e}_1\}|}{\|\mathbf{Sc}\{X_k^{k+1} \mathbf{e}_1\}\|_{L_2(S)}^2} = \frac{|\mathbf{Sc}\{Y_k^{k+1} \mathbf{e}_1\}|}{\|\mathbf{Sc}\{Y_k^{k+1} \mathbf{e}_1\}\|_{L_2(S)}^2} \leq \frac{2}{\pi} \frac{1}{2^n (n+1)!}.$$

Consequently, we have proved:

$$\begin{aligned}|X_k^{k+1,*} \mathbf{e}_1| |\alpha_k^{k+1}| \sqrt{2k+3} &\leq \frac{2}{\sqrt{3\pi}} \frac{r^k}{k!} \left((1-\delta) - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right) \\ |Y_k^{k+1,*} \mathbf{e}_1| |\beta_k^{k+1}| \sqrt{2k+3} &\leq \frac{2}{\sqrt{3\pi}} \frac{r^k}{k!} \left((1-\delta) - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right).\end{aligned}$$

With the previous inequalities we get

$$|\tilde{f}_2| = |f_2| \leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 + \frac{4}{\sqrt{3\pi}} \left((1-\delta) - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right) \sum_{n=1}^{\infty} \frac{r^n}{n!}.$$

Finally, we end with

$$|f_2| \leq 1 - \delta \implies \frac{4}{\sqrt{3\pi}} \sum_{n=1}^{\infty} \frac{r^n}{n!} \leq 1,$$

and, the last series is convergent for $r < 0.56$. Finally,

$$\sum_{n=0}^{\infty} \sqrt{2n+3} r^n \left| \left\{ X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right\} \right| < 1$$

converges for $0 \leq r < 0.05$. \blacksquare

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